GAUSSIAN PROCESSES
EXERCISE SHEET 2: LINDEBERG EXCHANGE PRINCIPLE AND HEAVY-TAILED CLT

Exercise 1 (Convergence in Law)**.** *Let $(X_n)_{n\geq 1}$ be a sequence of random variables and let X be a random variable. Suppose that for every smooth, compactly supported function f with bounded first three derivatives, we have*

$$\mathbb{E}[f(X_n)] \longrightarrow \mathbb{E}[f(X)] \quad \text{as } n \to \infty.$$

Show that this implies $X_n \xrightarrow{d} X$.

Exercise 2 (Truncation of Summands)**.** *Let $X_1, X_2, \dots$ be i.i.d. centered random variables with variance 1. Fix $\delta > 0$. For each k, define the truncated variables*

$$Y_k := X_k \mathbf{1}_{\{|X_k| \leq \delta n^{1/2}\}},$$

where $n \in \mathbb{N}$. Show that for any continuously differentiable function f with bounded first derivative, there exists a constant $c > 0$ such that, for all sufficiently large n,

$$\left| \mathbb{E}\Big[f\big(n^{-1/2}(X_1 + \cdots + X_n)\big) - f\big(n^{-1/2}(Y_1 + \cdots + Y_n)\big) \Big] \right| < c\,\delta.$$

Exercise 3 (Heavy-Tailed CLT)**.** *Consider a random variable X symmetric about 0.*

(1) If we have

$$\mathbb{P}(X > x) = \frac{1}{2} x^{-\alpha}, \qquad x \geq 1,$$

for some $\alpha \in (0, 2)$. Let $X_1, X_2, \dots$ be i.i.d. copies of X. Using characteristic functions (or any other method), prove that

$$n^{-1/\alpha} \sum_{i=1}^{n} X_i \xrightarrow{d} S_\alpha,$$

where S_α denotes the symmetric α-stable law.

(2) Suppose $\alpha > 1$. Does the same conclusion hold if the tail behavior is given more generally by

$$\mathbb{P}(X > x) = \frac{1}{2} x^{-\alpha}(1 + o(1)) \quad \text{as } x \to \infty.$$

Exercise 4 (Existence of non-trivial limits)**.** *Let $X_1, X_2, \dots$ be i.i.d. symmetric random variables. Decide whether one can always find a sequence of normalizing constants $b_n > 0$ such that*

$$\frac{X_1 + \cdots + X_n}{b_n} \xrightarrow{d} Z$$

for some non-trivial random variable Z.

(Hint: Construct some super-heavy-tailed random variables.)